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A RAPID REVIEW OF THEOREMS BASED ON INNER PRODUCT SPACES & MODIFIED THEOREM BASED ON INNER PRODUCT SPACES

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Abstract
In Mathematics, inner product spaces generalize Euclidean vector spaces, in which the inner product is the dot product or scalar product of Cartesian coordinates. Inner product spaces of infinite dimension are widely used in functional analysis. Inner product spaces over the field of complex numbers are sometimes referred to as unitary spaces. An inner product is a generalization of the dot product. In a vector space, it is a way to multiply vectors together, with the result of this multiplication being a scalar. In this paper work we will discuss about the inner product, its basic concepts necessary for theorem's proof and various theorems based on inner product spaces on real fields. Further, we modified the theorem of inner product spaces based on the symmetric and positive definite matrix associated fields real.
Keywords: [inner product spaces,matrices, norm, vector space]

1. INTRODUCTION

In Mathematics, an inner product spaces or, rarely a Hausdorff pre-Hilbert spaces is a zeal vector space or a complex vector space with an operation called an inner product. The inner product of two vectors in the space is a scalar, often denoted with angle brackets such as in$< a, b >$ inner product allows formal definition of intuitive Geometric notation, such as length, angle and orthogonality (zero inner product) of vectors[1,2] . Inner Product Spaces generalize Euclidean vector spaces, in which the inner product is the dot product or scalar product of cartesian coordinates. Inner Product Spaces of infinite dimensions are widely used in functional analysis. Inner Product Space over the field of complex numbers are sometimes referred to as unitary spaces. The first usage of the concept of a vector spaces with an inner product is due to Giuseppe peano in 1889 [3].An inner product naturally induces an associated norm, (denoted$|x|$ and $|y|$ so ,every inner product is a norm vector space. If this normed space is also complete (that is, a Banach space) then the inner product spaces is a Hilbert space. If an inner product space H is is not a Hilbert space, it can be extended by completions to a Hilbert spaceH'. This means that H is a linear subspace of $H', the\ inner$ product of H is the restoration of that of H',and H is dense in H' for the topology defined by the norm [1,4].

Inner Product Spaces

The inner product or dot product of $\mathrm{R^n}$ is a function $<,>$ defined as:

$$< u, v > = a_1b_1 + a_2b_2 + a_3b_3 + - - - - - - + a_nb_n$$

Where $u = [a_1, a_2, a_3, \dots\dots . a_n]^T, \in R^n$ and $v = [b_1, b_2, b_3, \dots\dots\dots, b_n]^T \in R^n$.

The inner product $<,>$satisfies the following properties:

- Linearity property: $< au + bv, w >= a < u, w > +b < v, w >$
- Symmetric property:$< u, v > =< v, u >$
- Positive definite property: $u \epsilon V, < u, u > \geq 0$ $and < u, u > = 0$ $if and only if u = 0$.

Then$(<,>, v)$ is Called inner product spaces on $\mathrm{R^n}$. [5]

Representation of InnerProduct

Let V be an n-dimensional vector space with an inner product < ,>, and let A be the matrix of < , > relative to a basis B. Then for any vectors$u, v \in V, < u, v >= xTAy$, where x and y are the coordinate vectors of u and v, respectively, i.e., x = [u]B and y = [v]B. [5,6]

Example 1. For the inner product of R3 defined by $< x, y = 2x_1y_1 - x_1y_2 - x_2y_1 + 5x_2y_2$, where x, its matrix relative to the standard basis $E = e_1, e_2$ is$A = \begin{bmatrix} \langle e_1, e_1\rangle & \langle e_1, e_2\rangle \\ \langle e_2, e_1\rangle & \langle e_2, e_2\rangle \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -1 & 5 \end{bmatrix}$

The inner product can be written as

$$\langle x, y\rangle = x^T A y = [x_1, x_2]\begin{bmatrix} 2 & -1 \\ -1 & 5 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

2. THEOREM BASED ON INNER PRODUCT SPACES

Theorem 2.1: Let A be a real positive definite matrix. Then the function $< x, y > = x^T A y$ is an inner product on R^n.[8]

Proof:

Suppose A is positive definite matrix

Then we have to show that $< x, y > = x^T A y$ is an inner product on R^n

First of all, note that we can write $< x, y > = x^T A y = x.(Ay)$

Where the "dot" is the dot product of R^n

Thus, $< x, y >$ is a real number.

We verify three properties of an inner product

Also, since we know that 'the dot product is commutative'.

We have:- $< x, y > = x.(Ay)$

Or$< x, y > = (Ay).x = (Ay)^T x$ [since, dot product is commutative and real number]

Or $< x, y > = y^T A^T x = y^T A x$ [by properties of transpose, A is symmetric]

Or $< x, y > = < y, x >$ [by definition of theorem]

Where, $x\ , y\ \epsilon V(vectorspace)$.

Thus the function<x, y> is symmetric.

And, for all $x, y, z\ \epsilon V$ then $< x + y, z > = (x + y)^T Az$

Or $< x + y, z > = (x^T + y^T)Az$ [using definition of theorem]

Or $< x + y, z > = x^T Az + y^T Az$

0r$< x + y > = < x, z > + < y, z >$ [by definition of theorem]

Thus,the $<$ x, y $>$ $is\ linear$

Also,if x is non-zero vector in R^nthen we have: $< x, x > = x^T Ax > 0$

Since, A is positive definite matrix then A can never zero

Also we have :$< 0,0 > = 0^T A0 = 0$

Thus,$< u, u > = 0\ iff\ u = 0$

Then we get $<$ x , y $>$ $is\ po$gitive definite matrix

Hence, the function $< x, y > = x^T Ay$ is an inner product on R^n.

Proof of the Theorem 2.1 by an example [9]

Suppose $A = In \times n$ be n×n order identity matrix, it all diagonal entries are positive and thedeterminant is positive so A is positive definite matrix where,

$u = (x_1, x_2, x_3$ ____ $, x_n)T$columns spaces belongs to Rn

And $v = (y_1, y_2, y_3, - - -, y_n)T$ columnspacesbelongstoRn

And $w = (w_1, w_2, w_3, - - -, w_n)$ column spaces belongs to Rn

Linear property: $< u + w, v > = (u + w)Av$ [By definition of theorem]

Or $< u + w, v > = [(x_1, +w_1, x_2 + w_2, x_3 + w_3,$ ____ $, x_n + w_3)T]T\ In \times n(y_1, y_2, y_3,$ ____ $y_n)T$

Or $< u + w, v > = (x_1 + w_1, x_2 + w_2, x_3 + w_3, - - -, x_n + w_n)n \times n(y_1, y_2, y_3, - - -, y_n)T$

Or $< u + w, v > = (x_1 + w_1, x_2 + w_2, x_3 + w_3,$ ____ $, n + w_n)(y_1, y_2, y_3,$ ____ $y_n)T$

Or $< u + w, v > = x_1y_1 + w_1y_1 + x_2y_2 + w_2y_2 + x_3y_3 + w_3y_3 + \_\_\_\_\_ + x_ny_n + w_ny_n$

Or $< u + w, v > = (x_1y_1 + x_2y_2 + x_3y_3 + \_\_\_\_\_\_ + x_ny_n) + (w_1y_1 + w_2y_2 + w_3y_3 + \_\_\_ + w_ny_n)$

Or, $< u + w, v > = < u, v > + < w, v >$

Symmetric Property: $< u, v > = uTAv$ [by theorem]

Or $< u, v > = \{(x_1, x_2, x_3, \_\_\_\_\_\_\_\_\_, x_n)T\}T\ In \times n(y_1, y_2, y_3, \_\_\_\_\_\_\_\_, y_n)T$

Or $< u, v > = (x_1, x_2, x_3, \_\_\_\_\_\_\_\_\_, x_n)n \times (y_1, y_2, y_3, \_\_\_\_\_\_\_\_, y_n)T$ [by pro. of transpose]

Or $< u, v > = (x_1, x_2, x_3, \_\_\_\_\_\_\_\_\_, x_n)(y_1, y_2, y_3, \_\_\_\_\_\_\_\_, y_n)T$

Or $< u, v > = x_1y_1 + x_2y_2 + x_3y_3 + - - - + x_ny_n$ ---------- (a)

Also $< v, u > = vTAu$ [by def. of theorem]

Or $< v, u > = \{(y_1, y_2, y_3, \_\_\_\_\_\_\_\_, y_n)\}In \times n(x_1, x_2, x_3, \_\_\_\_\_\_\_\_\_, x_n)T$

Or $< v, u > = (y_1, y_2, y_3, \_\_\_\_\_\_\_\_, y_n)n \times (x_1, x_2, x_3, \_\_\_\_\_\_\_\_\_, x_n)T$

Or $< v, u > = (y_1, y_2, y_3, \_\_\_\_\_\_\_\_, y_n)(x_1, x_2, x_3, \_\_\_\_\_\_\_\_\_, x_n)$

Or $< v, u > = x_1y_1 + x_2y_2 + x_3y_3 + - - - + x_ny_n$ ------------ (b)

Now, from equation (a) and (b) we get:

$< u, v > = < v, u >$

Positive Definite Property: $< u, u > = uTAu \geq 0$

If $< 0,0 > = 0TIn \times n0$

Or $< 0,0 > = 0$

Thus, we say that $< u, u > = 0 \in u = 0$

3. MODIFIED THEOREM BASED ON INNER PRODUCT SPACES

Theorem3.1 Let a (real) symmetric matrix A be positive definite then the function $< u, v > = \mathbf{u}^{\mathbf{T}}(\mathbf{A}\mathbf{A}^{\mathbf{T}})\mathbf{v}$ is an inner product on R^n.

Proof: Suppose $A_1 = AA^T$

Taking transpose in both sides, $A_1^T = (AA^T)^T$

Or $A^T = A^TA$ [since A is symmetric then $AA^T = A^TA$]

Or $\mathrm{A}_1^{\mathrm{T}} = \mathrm{AA}^{\mathrm{T}}$

Or A_1 is symmetric matrix ----------------- (i)

Also, since A is positive definite, then A^T is positive definite

Also, product of two positive definite matrices is always positive definite matrix. Thus,

$A_1 = AA^T$ is also positive definite matrix ----------------- (ii)

From equation (i) and (ii) we get:

A_1 is symmetric and positive definite ----------------- (ii)

Now, we have to show that $< u, v > = uTA_1v$ is an inner product on Rn

Linear property: Let u, v and w belongs to V then

$< u + w, v > = (u + w)^TA_1v$ [by def. Of theorem and using (iii)]

Or $< u + w, v > = (u^T + w^T)A_1v$ [using pro. Of transpose]

Or $< u + w, v > = u^TA_1v + w^TA_1v$

Or $< u + w, v > = < u, v > + < w, v >$

Symmetric property: Since $< u, v > = u^TA_1v$ [matrix representation]

Or $< u, v >^T = (u^TA_1v)^T$ [taking transpose in both sides]

Or $< u, v > = v^T{A_1}^T(u^T)^T$ [by pro. of transpose]

Or $< u, v > = v^TA_1u$ [using (i), since (uT)T = u]

Or $< u, v > = < v, u >$ [by def. of theorem]

Positive definite property: Since $< u,u >= u^T A_1 u \geq 0$

If$< 0,0 >= 0^T A_1 0$[by def. of property]

Or $< 0,0 >= 0$ [since, A1 is positive definite then A1is not equal to zero]

Or$< u,u >= 0 \Leftrightarrow u = 0$

Proof of the Theorem(3.1) by an example

Let $A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$ [Symmetric and positive definite matrix]

Or $A^T = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$ [Symmetric and positive definite matrix]

Or $A + A^T = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} + \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$

Or $A + A^T = \begin{bmatrix} 4 & 0 \\ 0 & 6 \end{bmatrix}$ [Symmetric and positive definite matrix]

Also, suppose $A_2 = \begin{bmatrix} 4 & 0 \\ 0 & 6 \end{bmatrix}$ ----------------- (a)

Now, we have to show that$\langle u, v \rangle = u^T A_2 v$ is also an inner product on R^2

For this, we take

$u = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $in\ R^2,\ v = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$ $in\ R^2\ and\ w = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ $in\ R^2$, whereR^2 is 2-dimensional Euclidean Space

Linear property:$< u + w,v >= (u + w)^T A_2 v$[by def. Of theorem and using (a)]

Or $\langle u + w,v \rangle = \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} -1 \\ 1 \end{bmatrix}\right)^T \begin{bmatrix} 4 & 0 \\ 0 & 6 \end{bmatrix}\begin{bmatrix} 0 \\ 2 \end{bmatrix} = \left(\begin{bmatrix} 1-1 \\ 0+1 \end{bmatrix}\right)^T \begin{bmatrix} 4 & 0 \\ 0 & 6 \end{bmatrix}\begin{bmatrix} 0 \\ 2 \end{bmatrix}$

Or $\langle u + w,v \rangle = \left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)^T \begin{bmatrix} 4 & 0 \\ 0 & 6 \end{bmatrix}\begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \end{bmatrix}\begin{bmatrix} 4 & 0 \\ 0 & 6 \end{bmatrix}\begin{bmatrix} 0 \\ 2 \end{bmatrix}$

Or $\langle u + w,v \rangle = \begin{bmatrix} 0 & 6 \end{bmatrix}\begin{bmatrix} 0 \\ 2 \end{bmatrix} = 0.0 + 6.2$

Or $\langle u + w,v \rangle = 12$ ----------------- (b)

Now $\langle u, v \rangle = u^T A_2 v$[by def. of theorem]

Or $\langle u, v \rangle = \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)^T \begin{bmatrix} 4 & 0 \\ 0 & 6 \end{bmatrix}\begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} 4 & 0 \\ 0 & 6 \end{bmatrix}\begin{bmatrix} 0 \\ 2 \end{bmatrix}$

Or $\langle u, v \rangle = \begin{bmatrix} 4 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 2 \end{bmatrix} = 4.0 + 0.2$

Or $\langle u, v \rangle = 0$ ------------------------ (c)

Now $\langle w, v \rangle = w^T A_2 v$[by def. of theorem]

Or $\langle w, v \rangle = \left(\begin{bmatrix} -1 \\ 1 \end{bmatrix}\right)^T \begin{bmatrix} 4 & 0 \\ 0 & 6 \end{bmatrix}\begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 & 1 \end{bmatrix}\begin{bmatrix} 4 & 0 \\ 0 & 6 \end{bmatrix}\begin{bmatrix} 0 \\ 2 \end{bmatrix}$

Or $\langle w, v \rangle = \begin{bmatrix} -4 & 6 \end{bmatrix}\begin{bmatrix} 0 \\ 2 \end{bmatrix} = -4.0 + 6.2$

Or $\langle w, v \rangle = 12$ -------------------- (d)

From equation (c) and (d) we get

$\langle u,v \rangle + \langle w,v \rangle = 12$ ---------------- (e)

Hence, from equation (b) and (e) we get: $\langle u + w,v \rangle = \langle u,v \rangle + \langle w,v \rangle$

Symmetric property: Since $\langle u, v \rangle = u^T A_2 v$ [by def. of theorem]

Or $\langle u, v \rangle = 0$ [using equation (c)]

Also, $\langle v, u \rangle = v^T A_2 u$[by def. of theorem]

Or $\langle v,u \rangle = \left(\begin{bmatrix} 0 \\ 2 \end{bmatrix}\right)^T \begin{bmatrix} 4 & 0 \\ 0 & 6 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & 2 \end{bmatrix}\begin{bmatrix} 4 & 0 \\ 0 & 6 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix}$

Or $\langle v,u \rangle = \begin{bmatrix} 0 & 12 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = 0.1 + 12.0$

Or $\langle v,u \rangle = 0$ ------------------- (f)

From equation (c) and (f) we get, $\langle u, v\rangle = \langle v, u\rangle$

Positive definite property: Since $< u, u >= u^T A_2 u \geq 0$

Case (i) if u is not equal to zero i.e. $u \neq 0$

$$< u, u > = \left(\begin{bmatrix}1\\0\end{bmatrix}\right)^T \begin{bmatrix}4 & 0\\0 & 6\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}4 & 0\\0 & 6\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix}$$

Or $$< u, u > = \begin{bmatrix}4 & 0\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix} = 4.1 + 0.0$$

Or $$< u, u >= 4 > 0$$

Case (ii) if u is equal to zero i.e. $u = 0$

$$< u, u >=< 0, 0 >= 0^T A_2 = \left(\begin{bmatrix}0\\0\end{bmatrix}\right)^T \begin{bmatrix}4 & 0\\0 & 6\end{bmatrix}\begin{bmatrix}0\\0\end{bmatrix}$$

Or $$< u, u > = \begin{bmatrix}0 & 0\end{bmatrix}\begin{bmatrix}4 & 0\\0 & 6\end{bmatrix}\begin{bmatrix}0\\0\end{bmatrix} = \begin{bmatrix}0 & 0\end{bmatrix}\begin{bmatrix}0\\0\end{bmatrix}$$

Or $$< u, u > = 0.0 + 0.0 = 0$$

Thus, we get $$< u, u > = 0 \Leftrightarrow u = 0$$

4. RESULT& DISCUSSION

An inner product space $(V, \langle,\rangle)$ is a vector space V over F together with an inner product: a function $\langle,\rangle : V \times V \to F$ satisfying the following properties $\forall x, y, z \in V, \lambda \in F$,

Linear: $\langle \lambda x + y, z\rangle = \lambda\langle x, z\rangle + \langle y, z\rangle$

Symmetric: $\langle y, x\rangle = \langle x, y\rangle$

Positive-definite: $x \neq 0 \implies \langle x, x\rangle > 0$

As we know that a square matrix$A = [aij]$ of n×n order $n \geq 1$ is called symmetric matrices if

$a_{ij} = a_{ji}\, for\, all\, A = A^T$.

Also, since we know that A be a symmetric matrix i.e$A = A^T$then A is said to be positive definite matrices if for every non zero vector u in Rnsuch as $< u, Au >= u^T Au$

When we replace A by $A + A^T$then positive definite matrices becomes$< u, (A + A^T)u >= u^T(A + A^T)u > 0$

Also, we replace A by AA^Tthen positive definite matrices becomes$< u, AA^T u > = u^T(AA^T)u > 0$

At first, we take symmetric matrices then it helps us to make positive definite matrices because a positive definite matrix must be symmetric.

Since, if A be positive definite matrices then the function $< u, v >= u^T Av$ is an inner product on Rn.(acc. to the given theorem)

Now, if replace A by $(A + A^T)$ then it is also positive definite matrices and the function$< u, v > = u^T(A + A^T)v$ is also inner product onRn.

Also, we replace A by AA^Tthen it is also positive definite matrices and the function becomes $< u, v >= u^T AA^T v$ is an inner product onRn.

Because it is easily satisfying the following three properties of an inner product spaces such that:-

Linear property:- $< au + bw, v >= a < u, v > +b < w, v >$

Symmetric property:–$< u, v > = < v, u >$

Positive definite property:- $< u, u > \geq 0\, i\int < u, u = 0 \Leftrightarrow u = 0 \int or\, all\, u\, in\, Rn.$

So, by considering the above explanation, we conclude that modified theorems is also develop unique concept in the field of inner product on Rn.

5. CONCLUSION

A matrix representation of n – dimensional vector spaces or columns spaces of Rn then how to show that it is an inner product on Rn and also it‟s modified form is an inner product on Rn.

There is theorem based on the inner product space that is, Theorem (2.1) Statement: Let A be a real positive definite matrix. Then the function $< x, y >= x^T Ay$is an inner product onRn.

Which can be modified based on real field for symmetric matrix & Skew symmetric matrix of real field Rnas:

Theorem (3.1) Let a symmetric matrix A be positive definite then the function $< u, v >= u^T (AA^T)v$is an inner product on Rn.

Thus, the function$< u, v >= \mathrm{u}^\mathrm{T}\mathrm{A}\mathrm{v}$is matrix representation then we converted from A to$\mathrm{A} + \mathrm{A}^\mathrm{T}$ in the function, so modified function becomes$< u, v >= \mathrm{u}^\mathrm{T}(\mathrm{A} + \mathrm{A}^\mathrm{T})\mathrm{v}$where, $\mathrm{A}^\mathrm{T}$is transpose of A

The proof for the modified theorem can be verified and these can be helpful in further development of inner product space.

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