REVIEW OF LAGRANGE'S MULTIPLIER METHOD

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Abstract

The origin and genesis of Lagrange multiplier is reviewed in this research paper. Lagrange, a Turinese mathematician was the first who conceived the theory of multipliers as a basic mathematical approach in the framework of statics that extracted the general equations of equilibrium for problems with constraints or imposed conditions. He primarily studied the different problems of statics, connected with the principle of virtual velocities, which ultimately suggested him to introduce the concept of multipliers. Though many workers before Lagrange also have addressed solutions to such problems where the ideas were similar to that of Lagrange but none of them sustained the scientific base that finds the exact value-oriented solutions. The results of Euler, who studied the calculus of variations, had reached a significant approximation although the complete solution of these problems was being proved by Lagrange, who justified the conditions of equilibrium as a mathematical expressions and he was thus credited as the inventor of this mathematical theory-"Lagrange Multipliers".

Since the concept of multipliers is originated in the context of statics so it is important to review in brief about Lagrange statics, which is as under:-

Lagrange Statics:

Lagrange initiated the study of statics with his book, 'MecaniqueAnalytique' [1] in which he identified the three basic principles of statics:

(i) The principle of the lever;

(ii) The principle of the composition of forces.

(iii) The principle of virtual velocities.

However,these two principles don't find its connectivity with origin of multipliers so have not been discussed here. On the other hand,the third principle of virtual velocities plays a great role in the introduction of concept of multipliers which is discussed here as under.

The virtualvelocity of a given body refers to the velocity that is received in the first instant of movement just after its equilibrium condition is interrupted. This principle of virtual velocities thus stated that the forces are in the state of equilibrium if they are inversely proportional to their virtual velocities. Further, Lagrange put his remarks on the phenomenon of movement of force as defined by Galileo, to be the product of a force and the virtual velocity, subsequently multiplied by a constant, and this is now known as "power" or "virtual power" [3]. Wallis also described the same concept but the used the word "moment" for this. Lagrange refined the above ideas of Galileo and Wallis and stated this concept as;"A body would be in equilibrium only if the algebraic sum of their virtual powers becomes zero."He extended his theoretical concept by employing mathematical symbols and this symbolic transcription provided a clear-cut interpretation of his thought. Thus,the symbolic expression of principle of virtual velocities is stated as,"The body is in equilibrium if it justifies the following expression:"

$$P\frac{dp}{dt} + Q\frac{dq}{dt} = 0$$

Where, P and Q represents the forces applied to a material point; p and q are the directions of forces P and Q, respectively and

dp/dt and da/dt

represent the virtual velocities that the body received by respective forces P and Q.

Also, in the above expression, the quantity

P=dp/dt

represents the moment of the force P.

Extension of his book 'Mecanique Analytique', Lagrange refined the earlier mathematical expression by substituting *dp/dt* with *dp*, which justified the concept of substitution as under: Three forces P, Q, R are employed along the given lines of a subject, which are in equilibrium. Consider p, q, r etc as lines that represent the directions of the forces P, Q, R. Let *dp,dq,dr,* etc are the variations or the differences or differentials of these lines which occurred due to an arbitrary infinitesimal change in the position of different bodies. These differences expressed the spaces, which they can cover under the action of the respective forces P, Q, and R in their respective directions. Using this, Lagrange simplified the earlier expression which involved virtual velocities *dp/dt and da/dt* as just the differences *dp,dq,dr,*which are proportional to the virtual velocities of the respective forces P,Q and R. Hence, the equilibrium statement is expressed as:
Though in this expression, he eliminated the name of virtual velocities, still the name of the principle continued the same as "principle of virtual velocities." Indeed pdp+qda+rdr are actually the works that turns virtual, if the system is in equilibrium and for this reason it was even called principle of virtual works.

More refinement in the principle was made later where he applied the principle of the equilibrium of a system of points while analyzing the translational and rotational equilibrium conditions.He further analyzed the importance of reducing Mechanics to purely analytical operations, which did not involve any intuitive geometrical considerations, and this rationalized him to introduce the method of multipliers. In this way, Lagrange studied the concept of equilibrium under constraints and solved the problems in a simpler manner.
Further, the concept of multipliers was studied for two different subject conditions, which were:

a) Multipliers and Statics of the point, and
b) Multipliers and Statics of a rigid body.

In the first case, Lagrange examined the equilibrium of a material point or of a system of points which has constraints and the constrained equations are represented as

dL=0,µdM=0,etc

Lagrange mentioned these real numbers as "Quantities indetemine coefficients" or like etc.
To this constrained differentials equation, the moment of forces (as explained earlier) are added. The sum of all the moments of the forces added with the different differential functions represents the equilibrium condition as expressed by the following equation:

$$Pdp + Qdq + \cdots + \lambda dL + \mu dM \ldots. = 0$$

This general equation of equilibrium marked the balanced state in an orthogonal reference framework. However, when it is considered in the direction 'x', the equation becomes:

$$P\frac{dp}{dx} + Q\frac{dq}{dx} + \cdots + \lambda\frac{dL}{dx} + \mu\frac{dM}{dx} \ldots. = 0$$

This above equation stated that the reaction of any constraint is equated to a representative force; and and here represent the agent forces; and are the virtual works that are realized under the impact of the respective agent forces.
This method thus enabled the solution of constrained problems in a similar manner as of constrained free problems. However, this approach had a consequence that it did not give the analytical and numerical treatment of the

constrained equilibrium conditions. The above equilibrium equation thus exhibited difficulty in determining the values of , etc. But, Lagrange overcomes this difficulty by making use of constraints that allowed a system with equal number of equations and variables. The following example justifies this clarity of Lagrange reasoning:

Under the given conditions in the ordinary space, there are three unknown quantities and another unknown quantity and if two agent forces are executed on a constraint represented by equation , the following four equations represent equilibrium conditions:

$$P\frac{dp}{dx}+Q\frac{dq}{dx}+\lambda\frac{dL}{dx}=0,$$

$$P\frac{dp}{dy}+Q\frac{dq}{dy}+\lambda\frac{dL}{dy}=0,$$

$$P\frac{dp}{dz}+Q\frac{dq}{dz}+\lambda\frac{dL}{dz}=0,$$

$$L=0.$$

Now, this mathematical expression enables the solution of the problem of the equilibrium of a material point or a system of points while determining the maxima or minima of a function under constraints.

In the second case, the equilibrium conditions for a rigid body were justified by the mathematical expression. Here, Lagrange considered a rigid body of a mass M. This case differs from the material point (where the force acted only in the point of application) in a manner that it encountered forces in all points of the body and hence the variation produced in the rigid body could not be equivalent to that of *dp* (which is in case of point statics). Here, the infinitesimal variation of the position of the body under the influence of agent forces is denoted as *dp* Lagrange obtained the equilibrium conditions of such rigid body by considering the following example:

A given mass of a body is considered as a set of infinite continuous points and as per the infinitesimal calculus, the total mass was equivalent to the sum of all these elements. If are the forces that act on the body, their “moments” or “virtual works” can be represented as:

$$pdp+qdq$$

equilibrium is achieved when variation of potential becomes zero, which are expresses as given equation: $S\,(pdp+qdq)=0$

Lagrange used the letter that marked the definite integral and notation *f* for the indefinite integral. Also *dm* denotes the infinitesimal element mass. The above equation is now generalized as:

$$f(pdp+qdq)=0$$

This equation represented the equilibrium if there do not exit any constraints.

However, under the conditions where we have constraints $L=0,M=0,etc.),$ then also the conditions ($\lambda dL=0,\mu dM=0,etc.$) are justified. and are multiplicative constants as defined in the above case of single point or system of points statics. The expression is thus written as: $S(\lambda\partial L+\mu\partial M+\quad)$

In this integral expression, the differentials are not speci□ed since they are dependent upon the variables of which *L,M,….*are functions. Thus, simple integral represents that the expression has only one variable and double integral will represents two variables and so on.

Moreover, in a particular context, there are chances that besides the forces that act on the whole mass and the reaction of the constraints, there exert forces on single points which is further subjected to constraints, then the general equilibrium equation becomes:

$$S(P\partial p+Q\partial q)dm+S(\lambda\partial L+\mu\partial M+\quad)$$
$$+P'\partial p'+Q'\partial q'+......+\alpha\partial A+\beta\partial B+\quad=0$$

Here, *P',Q'*,etc. represents the forces exhibited on single points and A=0,B=0, are respective constraints.

Lagrange thus determined the conditions of equilibrium for a rigid body and solved the problem as a generalized calculus of variations that locate the minima and maxima of a problem. This landmark contribution of Lagrange in calculus enables the solution of many problems. Even in non-linear programming, where the optimization of problems is difficult to interpret with other traditional approaches, Lagrange multiplier method finds its application to solve these problems. The basic problem in non-linear problems is that the non-linear constraints form feasible regions, which are difficult to locate, and the non-linear objectives contain local minima, which need to be trapped through some descent search methods. Lagrange multipliers solve these optimization problems by escaping the local minima while handling the non-linear constraints. Hence, Lagrange multiplier methods solve the equilibrium problems by determining the sufficient conditions as a function of minimum and maximum

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