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EXPLORING THE LEPTON MIXING PHENOMENON WITH STERILE NEUTRINOS

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Abstract
Ever since its inception by Pauli in order to explain the continuous beta decay spectrum, neutrinos have always posed to be a fascinating puzzle for particle physicists. Despite significant advances on experimental and theoretical fronts, several questions still remain unanswered in the neutrino sector. One of the foremost amongst these is the explanation of lepton mixing phenomenon, generally described through the Pontecorvo- Maki-Nakagawa-Sakata (PMNS) matrix, and a precise measurement of its parameters. The problem is further complexified by the observation of anomalies at several Short Base Line (SBL) neutrino experiments, hinting at the possibility of existence of additional sterile neutrinos. In this context, the present paper aims to discuss the lepton mixing phenomenon and the PMNS matrix in detail along with some of its popular parametrizations. Further, we would be providing a detailed mathematical discussion regarding the formulation of PMNS matrix in one of its most popular parametrizations, viz. the standard parametrization, for the standard three neutrino framework as well the 3+1 sterile neutrino scenario.
Keywords: Neutrinos, PMNS matrix, Sterile Neutrinos

1 INTRODUCTION

Neutrinos, proposed as the desperate remedy for continuous β decay spectrum, have turned out to be quite fascinating particles for the physicists. One of the most path breaking experimental milestone in this regard has been the observations of the neutrino oscillations involving solar,atmospheric,accelerator and reactor (anti)- neutrino at Super-Kamiokande (SK) [1] and Sudbury Neutrino Observatory (SNO) experiments [2]. These experiments provided a smoking gun signal for at least two neutrinos being massive and hence one of the most prominent motivation to look for the physics beyond Standard Model (SM). Even after decades of the continuous efforts on the theoretical as well as experimental fronts, number of questions regarding neutrinos still remained unanswered. Some prominent ones amongst these being- nature of neutrinos being Dirac or Majorana, possibility of CP violation in lepton sector parallel to the quark sector, the absolute neutrinos mass scale, and a precise measurement of lepton mixing parameters etc. The puzzle got further complicated by the observations of some anomalies at various Short Base Line (SBL) experiments, such as Liquid Scintillator Neutrino Detector (LSND) experiment (about 3.8σ) [3], Reactor experiments- Daya Bay and Reactor Experiment for Neutrino Oscillation (RENO) (about 2.8σ) [4, 5] and Gallium Solar Neutrino experiment (about 2.9σ) [6]. These anomalies hinted at the possibility of the existence of the new neutrino species, viz. the sterile neutrino, in the addition to the three active neutrinos (ν_e, ν_μ, ν_τ). The existence of massive neutrinos hinted at the possibility of lepton mixing parallel to the quark sector. In fact, the study of the lepton mixing phenomenon can indeed be considered as quite fruitful as many of unanswered questions regarding neutrinos lie in the masses and mixing sector.

The lepton mixing phenomenon can be best understood in terms of Pontecorvo-Maki-Nakawaga-Sakata (PMNS) matrix. This PMNS matrix essentially provides a relationship between the flavor eigenstates and the mass eigenstates of the leptons. Over the past few decades, many parametrizations of the PMNS matrix have been proposed in literature [7, 8, 9, 10, 11, 12]. In the present paper, we briefly discuss some of the most popular ones amongst these. Special emphasis would be laid on the Standard Parameterization proposed by Chau-Keung [7]. A detailed derivation of the formulation of the PMNS matrix in this parameterization for three active neutrinos as well as 3+1 sterile neutrino scenario would be presented.

This paper is organised as follows. In Section 2, we discuss about lepton mixing matrix and some of the parameterizations schemes. In Section 3, we will be formulating of mixing matrix for three active neutrinos and 3+1 sterile neutrinos through one of the most common parameterization, viz. Standard Parameterization. Finally, Section 4 summarizes our conclusions.

2 Lepton Mixing Matrix

As discussed earlier, the quest for lepton mixing parallel to that in quark sector stemmed from the signals of the massive neutrinos provided by various neutrino oscillation experiments over the last couple of decades. Before going into technical details, let us first give a simplistic explanation for the term 'lepton mixing'. Neutrinos of definite flavor that are produced in any charged-current interactions, have the probability to be partly in different flavor state when observed after some time. If the two states have the same masses, then both of them will propagate the same way and there will be no oscillations. If the two states have different masses but the flavor states are the same as the physical states (no mixing), then the flavor states will have definite evolution and they will not change to the other state. Thus, main criterion for the neutrino oscillation is that there should be a mass difference between the different neutrino states and the mass eigenstates are different from the flavor eigenstates [16]. This relationship is essentially given by the lepton mixing matrix . Inspired by koan-antikaon oscillation and the discovery of νμ, the idea of the lepton flavor mixing was advanced by Pontecorvo, Maki, Nakagawa and Sakata, who formulated the lepton mixing matrix which is named as Pontecorvo-Maki-Nakagawa- Sakata (PMNS) matrix [17, 18] after them. This mixing is represented through the unitary transformation that relates the flavor to the mass eigenstates as the left-handed neutrinos as follows -

$$\begin{pmatrix}\nu_e\\ \nu_\mu\\ \nu_\tau\end{pmatrix} = U.\begin{pmatrix}\nu_1\\ \nu_2\\ \nu_3\end{pmatrix} = \begin{pmatrix}U_{e1} & U_{e2} & U_{e3}\\ U_{\mu1} & U_{\mu2} & U_{\mu3}\\ U_{\tau1} & U_{\tau2} & U_{\tau3}\end{pmatrix}.\begin{pmatrix}\nu_1\\ \nu_2\\ \nu_3\end{pmatrix} \quad (1)$$

Here U is the neutrino mixing matrix and this matrix could parameterized through many ways such as-

1. Exponential Parameteriztion - The unitary exponential parameterization for the neutrino mixing matrix was proposed by Strumia and Vissani [8]. This type of parameterization is essentially used when there is need for the comparative analysis for the different neutrino mixing data. This parameterization is studied as -

$$U_{exp} = \exp A \quad (2)$$

where,

$$A = A_o = \begin{pmatrix}0 & \lambda_1 & \lambda_3 e^{i\delta}\\ -\lambda_1 & 0 & -\lambda_2\\ -\lambda_3 e^{-i\delta} & \lambda_2 & 0\end{pmatrix}, \quad (3)$$

the anti-Hermitian form of the matrix A ensures the unitarity of the transforms by the mixing matrix Uexp. The parameter δ accounts for the CP violation and the parameters λi are responsible for the flavor mixing. The hierarchy in the exponential quark mixing matrix is based on the single parameter which does not hold for neutrinos. The advantage of exponential parameterization is that it allows easy factorization of the rational part, the CP-violating terms and possible Majorana term.

2. Bimaximal Parameterization- This parameterization [9] is similar to the Woflenstein parameterization for quarks, using the fact that the quark mixing is minimal. However, in the case of lepton mixing, the experimental data suggested that the mixing is maximal. Experiments like K2K and Super-K provided data for the maximal mixing of the νμ and ντ which was θatm ≈ 45°. These three mixing angles represent mixing between three generations of the neutrinos, θatm = θ1 (viz mixing between 2nd and 3rd), θchz = θ2 (viz mixing between 3rd and 1st) and θsol = θ3 (viz mixing between 1st and 2nd). According to the global analysis of the neutrino oscillation experimental data, the elements of the modulus of the neutrino mixing matrix are-

$$|V| = \begin{pmatrix}0.77-0.88 & 0.47-0.61 & <0.20\\ 0.19-0.52 & 0.42-0.73 & 0.58-0.82\\ 0.20-0.53 & 0.44-0.74 & 0.56-0.81\end{pmatrix} \quad (4)$$

To expand the matrix in powers of one of the non-diagonal elements becomes complex as the non-diagonal terms in case of neutrinos are large (except Ve3). So, the bimaximal mixing pattern was used for the expansion as two of the mixing angles are maximal. These can be given as -

$$U_{BM} = \begin{pmatrix}1/\sqrt{2} & 1/\sqrt{2} & 0\\ -1/2 & 1/2 & 1/\sqrt{2}\\ 1/2 & -1/2 & 1/\sqrt{2}\end{pmatrix} \quad (5)$$

3. Tri-bimaximal Parameterization - There is another mixing pattern known as the Tri-Bimaximal mixing (TBM) [10, 11] which is much consistent with the experimental data. Therefore, it can be considered as one of the best approximation to the neutrino mixing matrix, three mixing angles are 45°, 0° and 35.3°. It is given by

$$U_{TB} = \begin{pmatrix}\sqrt{2/3} & 1/\sqrt{3} & 0\\ -1/\sqrt{6} & 1/\sqrt{3} & 1/\sqrt{2}\\ 1/\sqrt{6} & -1/\sqrt{3} & 1/\sqrt{2}\end{pmatrix}$$

However, the recent observations from several experiments like Daya Bay and RENO collaboration reported a non-vanishing mixing angle θ13 which requires modification in the TBM pattern [19].

4. Golden ratio mixing I and Golden ratio mixing II - This type of mixing was proposed as the solar neutrino mixing speculated the correlation of solar neutrino mixing angle with golden ratio as [12]

$$\psi = \psi^2 - 1 = \frac{1}{2}(1 + \sqrt{5}) \quad (7)$$

Two proposals were made- the first one is-

$$cot\theta_{12} = \psi \Rightarrow sin^2\theta_{12} = \frac{1}{1+\psi^2} = \frac{2}{5+\sqrt{5}} \simeq 0.276 \quad (8)$$

and the second one is-

$$cos\theta_{12} = \frac{\psi}{2} \Rightarrow sin^2\theta_{12} = \frac{1}{4}(3-\psi) = \frac{5-\sqrt{5}}{8} \simeq 0.345 \quad (9)$$

These mixings predict a vanishing reactor mixing angle (□13) and a maximal atmospheric mixing angle (□23) and hence, require modification in order to satisfy the current data from neutrino oscillation experiments regarding small, yet non-zero □13[13].

5. Standard Parameterization - This parameterization [7] is one the most popular parameterization proposed by Particle Data Group and is similar to the parameterization of the Cabibbo-Kobayashi-Maskawa matrix VCKM for quarks [14, 15]. Standard Parameterization is based on the Chau-Keung(CK) scheme that was originally given for quark mixing, which expresses the mixing matrix by three angles □12, □13 and □23 and one CP-violating phase angle □. Here the mixing matrix can be written as a product of three rotation matrices where one of them involves a phase, e.g.,

$$V = R_{23}(\theta_3, \delta)R_{12}(\theta_1, 0)R_{23}(\theta_2, 0) \quad 10$$

where

$$R_{12}(\theta_2, 0)\begin{pmatrix} c & se^{i\phi} & 0 \\ -se^{-i\phi} & c & 0 \\ 0 & 0 & 1 \end{pmatrix}, R_{23}(\theta_3, \delta)\begin{pmatrix} 1 & 0 & 0 \\ 0 & c & se^{i\phi} \\ 0 & -se^{-i\phi} & c \end{pmatrix} \quad (11)$$

where Rij(□, □) denotes a unitary rotation in the ij-plane by the angle □ and the phase □. R31 can be similarly obtained by cyclic permutation. The allowed ranges for the parameters in this type of parameterization are 0 ≤ □ij ≤ π/2 and 0 ≤ □ ≤ 2π. The general form of parameterization is

$$V = R_{..}(,)R_{..}(,)R_{..}(,) \quad (12)$$

where dots represent the indices of the R's indicating the choice of plane of the rotation. Now there are 3 possible choices of indices, i.e., 12, 23 and 31. If we choose the middle matrix as 12 then we are not allowed to use the indices 12 anymore for left and right one because then we do not obtain the most general matrix. However, we may take the left and right to have both the indices 23 or 31, or let one of them have 23 and the other 31. The left and the right indices may be taken as equal because rotation matrices do not commute. Thus, there are 4 possibilities with 12 in the middle, repeating the same procedure with 23 and 31, we get 3x4=12 such possibilities. Furthermore, we can put the phase in the three different slots. Therefore, in total we have 3X12=36 different looking parameterizations which are all equivalent to KM parameterization. So the CK parameterization that we use is one of the form and is given by

$$V = R_{23}(\theta_y, 0)R_{31}(-\theta_z, \phi)R_{12}(\theta_x, 0). \quad (13)$$

3 Formulation of mixing matrix

Out of the various parametrizations discussed above, the Standard Parameterization is one of the most popular ones and is widely used in literature [23, 24]. Therefore, the present section focuses on the detailed mathematical discussion on the formulation of the mixing matrix. In this, assuming the neutrinos to be Majorana fermions. In this section our focus is on the formulation of the mixing matrix through Standard Parameterization method when neutrinos are considered to be of Majorana type. The PMNS matrix which links flavor eigenstates □e, □μ, □□ to the mass eigenstates □1, □2, □3, cab be given as -

$$\begin{pmatrix} \nu_e \\ \nu_\mu \\ \nu_\tau \end{pmatrix} = \begin{pmatrix} V_{e1} & V_{e2} & V_{e3} \\ V_{\mu 1} & V_{\mu 2} & V_{\mu 3} \\ V_{\tau 1} & V_{\tau 2} & V_{\tau 3} \end{pmatrix} \begin{pmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \end{pmatrix} \quad (14)$$

If the neutrinos are of Dirac type, the neutrino mixing matrix can be written as follows (with three mixing angles and a Dirac CP-violating phase, analogous to that of quarks), viz.,

$$V = \begin{pmatrix} c_2c_3 & c_2s_3 & s_2e^{-i\delta} \\ -c_1s_3 - s_1s_2c_3e^{i\delta} & c_1c_3 - s_1s_2s_3e^{i\delta} & s_1c_2 \\ s_1s_3 - c_1s_2c_3e^{i\delta} & -s_1c_3 - c_1s_2s_3e^{i\delta}e^{i\delta} & c_1c_2 \end{pmatrix} \quad (15)$$

where si = sin□i, ci = cos□i (for i=1,2,3), and □ is the Dirac CP-violating phase. The scenario changes when

the neutrinos are Majorana type. In this case, the mixing matrix can be given as -
V = U.P (16)
a product of the Dirac neutrino mixing matrix and a diagonal phase matrix with two unremovable phase angles diag(ei, ei, 1), where , are the Majorana CP-violating phases. The Dirac CP-violating phases is associated with the neutrino oscillations, CP and T violation, and the Majorana CP-violating phases are associ- ated with the neutrinoless-double beta decay and lepton-number-violating processes. The Dirac mixing matrix is obtained by the product of rotation matrices. The structure of the complex rotation matrices are given by [21]

$$R_{ij} = \begin{bmatrix} C_{ij} & Si_{ij} \\ -S_{ij} & C_{ij} \end{bmatrix} \tag{17}$$

$$\tilde{R_{ij}} = \begin{bmatrix} C_{ij} & \tilde{S_{ij}} \\ -\tilde{S}^*_{ij} & C_{ij} \end{bmatrix} \tag{18}$$

$$C_{ij} \equiv cos\Theta_{ij}, S_{ij} \equiv sin\Theta_{ij} and \tilde{S_{ij}} \equiv S_{ij}e^{-}\phi_{ij}$$

3 Active Neutrinos

In the case of three 3 Majorana neutrinos, 3X3 mixing matrix is analogous to the CKM matrix for the quarks. However, due to the Majorana nature of the neutrinos, it depends on the six independent pa- rameters- three mixing angles and three phases. In this case the mixing matrix is paramterized as [22]

$$V = U.P \tag{19}$$

where,

$$U = R_{23}\tilde{R_{13}}R_{12} \tag{20}$$

and

$$P = diag(e^{i\alpha}, e^{i\beta}, 1) \tag{21}$$

the rotation matrices are obtained as-

$$R_{23} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} \\ 0 & -s_{23} & c_{23} \end{pmatrix} \tag{22}$$

$$\tilde{R_{13}} = \begin{pmatrix} c_{13} & 0 & s_{13}e^{-i\delta_{CP}} \\ 0 & 1 & 0 \\ -s_{13}e^{i\delta_{CP}} & 0 & c_{13} \end{pmatrix} \tag{23}$$

$$R_{12} = \begin{pmatrix} c_{12} & s_{12} & 0 \\ -s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{24}$$

$$\tilde{R_{13}}R_{12} = \begin{pmatrix} c_{13}c_{12} & c_{13}s_{12} & s_{13}e^{i\delta_{CP}} \\ -s_{12} & c_{12} & 0 \\ -c_{12}s_{13}e^{-i\delta_{CP}} & -s_{12}s_{13}e^{-i\delta_{CP}} & c_{13} \end{pmatrix} \tag{25}$$

$$R_{23}\tilde{R_{13}}R_{12} = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta_{CP}} \\ -s_{12}c_{23} - c_{12}s_{13}s_{23}e^{i\delta_{CP}} & c_{12}c_{23} - s12s_{13}s_{23}i\delta_{CP} & c_{13}s_{23} \\ s_{12}s_{23} - c_{12}s_{13}c_{23}i\delta_{CP} & -c_{12}s_{23} - s_{12}s_{13}c_{23}i\delta_{CP} & c_{13}c_{2}3 \end{pmatrix} \tag{26}$$

The Neutrino mixing matrix for three active neutrinos case is:

$$V = U.P \tag{27}$$

$$V = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta_{CP}} \\ -s_{12}c_{23} - c_{12}s_{13}s_{23}e^{i\delta_{CP}} & c_{12}c_{23} - s12s_{13}s_{23}i\delta_{CP} & c_{13}s_{23} \\ s_{12}s_{23} - c_{12}s_{13}c_{23}i\delta_{CP} & -c_{12}s_{23} - s_{12}s_{13}c_{23}i\delta_{CP} & c_{13}c_{2}3 \end{pmatrix} . \begin{pmatrix} e^{i\alpha} & 0 & 0 \\ 0 & e^{i\beta} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$V = \begin{pmatrix} c_{12}c_{13}e^{i\alpha} & s_{12}c_{13}e^{i\beta} & s_{13}e^{-i\delta_{CP}} \\ -s_{12}c_{23} - c_{12}s_{13}s_{23}e^{i\delta_{CP}}e^{i\alpha} & c_{12}c_{23} - s12s_{13}s_{23}e^{i\delta_{CP}}e^{i\beta} & c_{13}s_{23} \\ s_{12}s_{23} - c_{12}s_{13}c_{23}e^{i\delta_{CP}}e^{i\alpha} & -c_{12}s_{23} - s_{12}s_{13}c_{23}e^{i\delta_{CP}}e^{i\beta} & c_{13}c_{2}3 \end{pmatrix} \tag{28}$$

3+1 Sterile scenario

In the 3+1 scenario, i.e. three active neutrinos and one sterile neutrino, the mixing matrix depends on 12 parameters viz., six mixing angles, three Dirac CP violating phases and three Majorana phases. Mathematically, it can be given as -

$$V = U.P \tag{29}$$

where,

$$U = R_{34}\tilde{R_{24}}\tilde{R_{14}}R_{23}\tilde{R_{13}}R_{12} \tag{30}$$

and

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & e^{-i\alpha/2} & 0 & 0 \\ 0 & 0 & e^{-i\beta/2+i\delta_{13}} & 0 \\ 0 & 0 & 0 & e^{-i\gamma/2+i\delta_{14}} \end{pmatrix} \tag{31}$$

here Rij and R˜ij represents a real and complex 4X4 rotation in the (i,j) plane, respectively containing the 2X2 sub matrices

$$R_{ij} = \begin{bmatrix} C_{ij} & S_{ij} \\ -S_{ij} & C_{ij} \end{bmatrix} \tag{32}$$

$$\tilde{R_{ij}} = \begin{bmatrix} C_{ij} & \tilde{S_{ij}} \\ -\tilde{S}^*_{ij} & C_{ij} \end{bmatrix} \tag{33}$$

where, Cij cos□ij, Sij sin□ij and S˜ij Sije−□ij
4X4 rotation matrices are :

$$R_{34} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & c_{34} & s_{34} \\ 0 & 0 & -s_{34} & c_{34} \end{pmatrix}, R_{12} = \begin{pmatrix} c_{12} & s_{12} & 0 & 0 \\ -s_{12} & c_{12} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{34}$$

$$R_{23} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & c_{23} & s_{23} & 0 \\ 0 & -s_{23} & c_{23} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tilde{R_{24}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & c_{24} & 0 & s_{24}e^{-i\delta_{24}} \\ 0 & 0 & 1 & 0 \\ 0 & -s_{24}e^{i\delta_{24}} & 0 & c_{24} \end{pmatrix} \tag{35}$$

$$\tilde{R_{13}} = \begin{pmatrix} c_{13} & 0 & s_{13}e^{-i\delta_{13}} & 0 \\ 0 & 1 & 0 & 0 \\ -s_{13}e^{i\delta_{13}} & 0 & c_{13} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, R_{14} = \begin{pmatrix} c_{14} & 0 & 0 & s_{14}e^{-i\delta_{14}} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -s_{14}e^{i\delta_{14}} & 0 & 0 & c_{14} \end{pmatrix} \tag{36}$$

The Dirac mixing matrix is obtained by -

$$R_{1\tilde{3}}R_{12} = \begin{pmatrix} c_{13}c_{12} & c_{13}s_{12} & s_{13}e^{-i\delta_{13}} & 0 \\ -s_{12} & c_{12} & 0 & 0 \\ -s_{13}c_{13}e^{i\delta_{13}} & -c_{13}s_{12}e^{i\delta_{13}} & c_{13} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{37}$$

$$R_{23}\tilde{R_{13}}R_{12} = \begin{pmatrix} c_{12}c_{13} & c_{13}s_{12} & s_{13}e^{i\delta_{13}} & 0 \\ -c_{23}s_{12} - s_{13}c_{12}s_{23}e^{i\delta_{13}} & c_{23}c_{12} - s_{13}s_{12}s_{23}e^{i\delta_{13}} & c_{13}s_{23} & 0 \\ s_{23}s_12 - s_13c_12c_23e^{i\delta_{13}} & s_23c_{12} - c_{23}s_13s_{12}e^{i\delta_{13}} & c_{23}c_{13} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{38}$$

$R_{14}R_{23}\tilde{R_{13}}R_{12} =$

$$\begin{pmatrix} c_{13}c_{12}c_{14} & c_{13}s_{12}c_{14} & s_{13}c_{14}e^{i\delta_{13}} & s_{14}e^{-i\delta_{14}} \\ -s_{12}c_{23} - s_{13}c_{12}s_{23}e^{i\delta_{13}} & c_{12}c_{23} - s_{13}s_{12}s_{23}e^{i\delta_{13}} & c_{13}s_{23} & 0 \\ s_{12}s_{23} - s_{13}c_{12}c_{23}e^{i\delta_{13}} & -c_{12}c_{23} - s_{13}s_{12}c_{23}e^{i\delta_{13}} & c_{13}c_{23} & 0 \\ c_{12}c_{13}s_{14}e^{i\delta_{14}} & -c_{13}s_{12}s_{14}e^{i\delta_{14}} & -s_{13}s_{14}e^{i\delta_{14}-i\delta_{13}} & c_{14} \end{pmatrix} \tag{39}$$

Finally, we obtain -

$$\tilde{R_{24}}\tilde{R_{14}}R_{23}\tilde{R_{13}}R_{12} = \begin{pmatrix} A & B & C & D \\ E & F & G & H \\ I & J & K & L \\ M & N & O & P \end{pmatrix} \tag{40}$$

where,
$A = c_{13}c_{12}c_{14}$
$B = c_{13}s_{12}c_{14}$
$C = s_{13}c_{14}e^{i\delta_{13}}$
$D = s14e^{-i\delta_{14}}$
$E = -c_{24}s_{12}c_{23} - c_{24}s_{13}c_{12}s_{23}e^{i\delta_{13}} - c_{13}c_{12}s_{14}s_{24}e^{i\delta_{24}+i\delta_{14}}$
$F = c_{24}c_{12}c_{23} - c_{24}s_{13}s_{12}s_{23}e^{i\delta_{13}} - c_{13}s_{12}s_{14}s_{24}e^{i\delta_{24}+i\delta_{14}}$
$G = c_{13}s_{23} - s_{13}s_{14}s_{24}e^{-i\delta_{13}+i\delta_{24}+i\delta_{14}}$
$H = c_{14}s_{24}e^{i\delta_{24}}$
$I = s_{12}s_{23} - s_{13}c_{12}c_{23}e^{i\delta_{13}}$
$J = -c_{12}c_{23} - s_{13}s_{12}c_{23}e^{i\delta_{13}}$
$K = c_{13}c_{23}$
$L = 0$
$M = -c_{13}c_{12}e^{i\delta_{14}}s_{14}c_{24} - e^{i\delta_{24}}s_{24}(-s_{12}c_{23} - s_{13}c_{12}s_{23}e^{i\delta_{13}})$
$N = -c_{13}s_{12}e^{i\delta_{14}}s_{14}c_{24} - e^{i\delta_{24}}s_{24}(c_{12}c_{23} - s_{13}s_{12}s_{23}e^{i\delta_{13}})$
$P = -c_{13}e^{i\delta_{14}-i\delta_{13}}$
$O = c_{14}c_{24}$
And,

$$U = R_{34}\tilde{R_{24}}\tilde{R_{14}}R_{23}\tilde{R_{13}}R_{12} = \begin{pmatrix} U_{e1} & U_{e2} & U_{e3} & U_{e4} \\ U_{\mu_1} & U_{\mu_2} & U_{\mu_3} & U_{\mu_4} \\ U_{\tau 1} & U_{\tau 2} & U_{\tau 3} & U_{\tau 4} \\ U_{s1} & U_{s2} & U_{s3} & U_{s4} \end{pmatrix} \tag{41}$$

where,
$U_{e1} = c_{13}c_{12}c_{14}$
$U_{e2} = c_{13}s_{12}c_{14}$
$U_{e3} = s_{13}c_{14}e^{i\delta_{13}}$

$U_{e4} = s14e^{-i\delta_{14}}$
$U_{\mu_1} = -c_{24}s_{12}c_{23} - c_{24}s_{13}c_{12}s_{23}e^{i\delta_{13}} - c_{13}c_{12}s_{14}s_{24}e^{i\delta_{24}+i\delta_{14}}$
$U_{\mu_2} = c24c12c23 - c24s13s12s23e^{i\delta_{13}} - c13s12s14s24e^{i\delta_{24}+i\delta_{14}}$
$U_{\mu_3} = c_{13}s_{23} - s_{13}s_{14}s_{24}e^{-i\delta_{13}+i\delta_{24}+i\delta_{14}}$
$U_{\mu_4} = c_{14}s_{24}e^{i\delta_{24}}$
$U_{\tau 1} = s34(-c13c12e^{i\delta_{14}}s14c24 - e^{i\delta_{24}}s24(-s12c23 - s13c12e^{i\delta_{13}}s23)) + c34(s12c34 - s13c12c23e^{i\delta_{13}})$
$U_{\tau 2} = s_{34}(-c_{13}s_{12}e^{i\delta_{14}}s_{14}c_{24} - e^{i\delta_{24}}s_{24}(-c_{12}c_{23} - s_{13}s_{12}e^{i\delta_{13}}s_{23})) + c_{34}(c_{12}c_{34} - s_{13}s_{12}c_{23}e^{i\delta_{13}})$
$U_{\tau 3} = c_{13}c_{34}c_{23} + s_{34}(-c_{13}e^{i\delta_{24}}s_{23}s_{24} - s_{13}e^{i\delta_{14}-i\delta_{13}}s_{14}c_{24})$
$U_{\tau 4} = c14s34c24$
$U_{s1} = c_{34}(-c_{13}c_{12}e^{i\delta_{14}}s_{14}c_{24} - e^{i\delta_{24}}s_{24}(-s_{12}c_{23} - s_{13}c_{12}e^{i\delta_{13}}s_{23})) - s_{34}(s_{12}c_{34} - s_{13}c_{12}c_{23}e^{i\delta_{13}})$
$U_{s2} = c_{34}(-c_{13}s_{12}e^{i\delta_{14}}s_{14}c_{24} - e^{i\delta_{24}}s_{24}(-c_{12}c_{23} - s_{13}s_{12}e^{i\delta_{13}}s_{23})) - s_{34}(c_{12}c_{34} - s_{13}s_{12}c_{23}e^{i\delta_{13}})$
$U_{s3} = -c13s34c23 + c34(-c13e^{i\delta_{24}}s23s24 - s13e^{i\delta_{14}-i\delta_{13}}s14c24)$
$U_{s4} = c_{14}c_{34}c_{24}$

The neutrino mixing matrix for 3+1 scenario is given by

$$\begin{pmatrix} V_{e1} & V_{e2} & V_{e3} & V_{e4} \\ V_{\mu 1} & V_{\mu 2} & V_{\mu 3} & V_{\mu 4} \\ V_{\tau 1} & V_{\tau 2} & V_{\tau 3} & V_{\tau 4} \\ V_{s1} & V_{s2} & V_{s3} & V_{s4} \end{pmatrix} = \begin{pmatrix} U_{e1} & U_{e2} & U_{e3} & U_{e4} \\ U_{\mu_1} & U_{\mu_2} & U_{\mu_3} & U_{\mu_4} \\ U_{\tau 1} & U_{\tau 2} & U_{\tau 3} & U_{\tau 4} \\ U_{s1} & U_{s2} & U_{s3} & U_{s4} \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & e^{-i\alpha/2} & 0 & 0 \\ 0 & 0 & e^{-i\beta/2+i\delta_{13}} & 0 \\ 0 & 0 & 0 & e^{-i\gamma/2+i\delta_{14}} \end{pmatrix} \tag{42}$$

Explicitly, the elements of the mixing matrix in this case turn out to be -
$V_{e1} = c_{13}c_{12}c_{14}$
$V_{e2} = c_{13}s_{12}c_{14}e^{-i\alpha/2}$
$V_{e3} = s_{13}c_{14}e^{-i\beta/2}$
$V_{e4} = s14e^{-i\gamma/2}$
$V_{\mu 1} = -c_{24}s_{12}c_{23} - c_{24}s_{13}c_{12}s_{23}e^{i\delta_{13}} - c_{13}c_{12}s_{14}s_{24}e^{i\delta_{24}+i\delta_{14}}$
$V_{\mu 2} = e^(-i\alpha/2)(c_{24}c_{12}c_{23} - c_{24}s_{13}s_{12}s_{23}e^{i\delta_{13}} - c_{13}s_{12}s_{14}s_{24}e^{i\delta_{24}+i\delta_{14}})$
$V_{\mu 3} = e^{i\delta_{13}-i\beta/2}(c13s23 - s13s14s24e^{-i\delta_{13}+i\delta_{24}+i\delta_{14}}$
$V_{\mu 4} = c_{14}s_{24}e^{-i\gamma/2+i\delta_{24}+i\delta_{14}}$
$V_{\tau 1} = s_{34}(-c_{13}c_{12}e^{i\delta_{14}}s_{14}c_{24} - e^{i\delta_{24}}s_{24}(-s_{12}c_{23} - s_{13}c_{12}e^{i\delta_{13}}s_{23})) + c_{34}(s_{12}c_{34} - s_{13}c_{12}c_{23}e^{i\delta_{13}})$
$V_{\tau 2} = e^(-i\alpha/2)(s34(-c13s12e^{i\delta_{14}}s14c24 - e^{i\delta_{24}}s24(-c12c23 - s13s12e^{i\delta_{13}}s23)) + c34(c12c34 - s13s12c23e^{i\delta_{13}}))$
$V_{\tau 3} = e^{i\delta_{13}-i\beta/2}(c_{13}c_{34}c_{23} + s_{34}(-c_{13}e^{i\delta_{24}}s_{23}s_{24} - s_{13}e^{i\delta_{14}-i\delta_{13}}s_{14}c_{24}))$
$V_{\tau 4} = c_{14}s_{34}c_{24}e^{-i\gamma/2+i\delta_{14}}$
$V_{s1} = c34(-c13c12e^{i\delta_{14}}s14c24 - e^{i\delta_{24}}s24(-s12c23 - s13c12e^{i\delta_{13}}s23)) - s34(s12c34 - s13c12c23e^{i\delta_{13}})$
$V_{s2} = e^{-i\alpha/2}(c_{34}(-c_{13}s_{12}e^{i\delta_{14}}s_{14}c_{24} - e^{i\delta_{24}}s_{24}(-c_{12}c_{23} - s_{13}s_{12}e^{i\delta_{13}}s_{23})) - s_{34}(c_{12}c_{34} - s_{13}s_{12}c_{23}e^{i\delta_{13}}))$
$V_{s3} = e^{i\delta_{13}-i\beta/2}(-c_{13}s_{34}c_{23} + c_{34}(-c_{13}e^{i\delta_{24}}s_{23}s_{24} - s_{13}e^{i\delta_{14}-i\delta_{13}}s_{14}c_{24}))$
$V_{s4} = c14c34c24e^{-i\gamma/2+i\delta_{14}}$.

3 Conclusions

The present paper is an attempt to provide a detailed discussion regarding the formulation of the PMNS matrix characterizing the lepton mixing phenomenon. Some of the most popular parameterizations of PMNS matrix have been briefly discussed. Out of these, the Standard Parameterization is the one which has gathered a lot of attention of the theoretical physicists in the last couple of decades. A detailed mathematical derivation of the PMNS matrix in this parameterization for the three active neutrinos as well as 3+1 sterile neutrino scenarios has been presented here. This, in turn, can turn out to be a stepping stone for theoretical particle physicists to formulate models for providing a thorough explanation of the lepton mixing phenomenon in order to unravel some of the fascinating yet mysterious aspects of neutrinos physics.

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